POINTWISE WEIGHTED APPROXIMATION OF FUNCTIONS WITH ENDPOINT SINGULARITIES BY COMBINATIONS OF BERNSTEIN OPERATORS

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Abstract. We give direct and inverse theorems for the weighted approximation of functions with endpoint singularities by combinations of Bernstein operators.

1. Introduction

The set of all continuous functions, defined on the interval I, is denoted by $C(I)$. For any $f \in C([0,1])$, the corresponding *Bernstein operators* are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x),$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ k = 0,1,2,\dots,n,\ x \in [0,1].$$

Approximation properties of Bernstein operators have been studied very well (see [2], [3], [5]-[8], [12]-[14], for example). In order to approximate the functions with singularities, Della Vecchia et al. [3] and Yu-Zhao [12] introduced some kinds of *modified Bernstein operators*. Throughout the paper, C denotes a positive constant independent of n and x, which may be different in different cases.
Let

$$w(x) = x^{\alpha}(1-x)^{\beta},\ \alpha,\ \beta \geqslant 0,\ \alpha+\beta > 0,\ 0 \leqslant x \leqslant 1.$$

and

$$C_w := \{f \in C((0,1)) : \lim_{x\longrightarrow 1}(wf)(x) = \lim_{x\longrightarrow 0}(wf)(x) = 0\}.$$

The *norm* in C_w is defined by $\|wf\|_{C_w} := \|wf\| = \sup_{0\leqslant x\leqslant 1} |(wf)(x)|$. Define

$$W^r_{w,\lambda} := \{f \in C_w : f^{(r-1)} \in A.C.((0,1)),\ \|w\varphi^{r\lambda} f^{(r)}\| < \infty\}.$$

For $f \in C_w$, define the *weighted modulus of smoothness* by

$$\omega^r_{\varphi^\lambda}(f,t)_w := \sup_{0<h\leqslant t}\{\|w\triangle^r_{h\varphi^\lambda} f\|_{[16h^2,1-16h^2]} + \|w\overrightarrow{\triangle}^r_h f\|_{[0,16h^2]} + \|w\overleftarrow{\triangle}^r_h f\|_{[1-16h^2,1]}\},$$

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where

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^{r}(-1)^k \binom{r}{k} f(x + (\frac{r}{2} - k)h\varphi(x)),$$

$$\overrightarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r}(-1)^k \binom{r}{k} f(x + (r-k)h),$$

$$\overleftarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r}(-1)^k \binom{r}{k} f(x - kh),$$

and $\varphi(x) = \sqrt{x(1-x)}$. Della Vecchia *et al.* firstly introduced $B_n^*(f,x)$ and $\bar{B}_n(f,x)$ in [3], where the properties of $B_n^*(f,x)$ and $\bar{B}_n(f,x)$ are studied. Among others, they prove that

$$\|w(f - B_n^*(f))\| \leqslant C\omega_\varphi^2(f, n^{-1/2}),\ f \in C_w,$$

$$\|\bar{w}(f - \bar{B}_n(f))\| \leqslant \frac{C}{n^{3/2}} \sum_{k=1}^{[\sqrt{n}]} k^2 \omega_\varphi^2(f, \frac{1}{k})_{\bar{w}}^*,\ f \in C_{\bar{w}},$$

where $w(x) = x^\alpha(1-x)^\beta,\ \alpha,\ \beta \geqslant 0,\ \alpha+\beta > 0,\ 0 \leqslant x \leqslant 1$. In [11], for any $\alpha,\ \beta > 0,\ n \geqslant 2r + \alpha + \beta$, there hold

$$\|wB_{n,r}^*(f)\| \leqslant C\|wf\|,\ f \in C_w,$$

$$\|w(B_{n,r}^*(f) - f)\| \leqslant \begin{cases} \frac{C}{n^r}(\|wf\| + \|w\varphi^{2r} f^{(2r)}\|), f \in W_w^{2r}, \\ C(\omega_\varphi^{2r}(f, n^{-1/2})_w + n^{-r}\|wf\|), f \in C_w. \end{cases}$$

$$\|w\varphi^{2r} B_{n,r}^{*(2r)}(f)\| \leqslant \begin{cases} Cn^r\|wf\|, f \in C_w, \\ C(\|wf\| + \|w\varphi^{2r} f^{(2r)}\|), f \in W_w^{2r}. \end{cases}$$

and for $0 < \gamma < 2r$,

$$\|w(B_{n,r}^*(f) - f)\| = O(n^{-\gamma/2}) \Longleftrightarrow \omega_\varphi^{2r}(f,t)_w = O(t^r).$$

Ditzian and Totik [5] extended this method of combinations and defined the following combinations of Bernstein operators:

$$B_{n,r}(f,x) := \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f,x).$$

with the conditions

(a) $n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Cn$,

(b) $\sum_{i=0}^{r-1} |C_i(n)| \leqslant C$,

(c) $\sum_{i=0}^{r-1} C_i(n) = 1$,

(d) $\sum_{i=0}^{r-1} C_i(n) n_i^{-k} = 0$, for $k = 1, \ldots, r-1$.

2. The main results

Now, we can define our *new combinations of Bernstein operators* as follows:

$$B_{n,r}^*(f,x) := B_{n,r}(F_n, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(F_n, x), \tag{2.1}$$

where $C_i(n)$ satisfy the conditions (a)-(d). For the details, it can be referred to [11]. Our main results are the following:

Theorem 2.1. *If* $\alpha,\ \beta>0$, *for any* $f\in C_w$, *we have*

$$\|wB^{*(r)}_{n,r-1}(f)\|\leqslant Cn^r\|wf\|. \tag{2.2}$$

Theorem 2.2. *For any* $\alpha,\ \beta>0,\ 0\leqslant\lambda\leqslant 1$, *we have*

(2.3)

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)|\leqslant\left\{\begin{array}{ll} Cn^{r/2}\{max\{n^{r(1-\lambda)/2},\varphi^{r(\lambda-1)}(x)\}\}\|wf\|, & f\in C_w,\\ C\|w\varphi^{r\lambda}f^{(r)}\|, & f\in W^r_{w,\lambda}.\end{array}\right.$$

Theorem 2.3. *For* $f\in C_w,\ \alpha,\ \beta>0,\ \alpha_0\in(0,r),\ 0\leqslant\lambda\leqslant 1$, *we have*

$$w(x)|f(x)-B^*_{n,r-1}(f,x)|=O((n^{-\frac{1}{2}}\varphi^{-\lambda}(x)\delta_n(x))^{\alpha_0})\Longleftrightarrow\omega^r_{\varphi^\lambda}(f,t)_w=O(t^{\alpha_0}). \tag{2.4}$$

3. Lemmas

Lemma 3.1. *([13]) For any non-negative real* u *and* v, *we have*

$$\sum_{k=1}^{n-1}(\frac{k}{n})^{-u}(1-\frac{k}{n})^{-v}p_{n,k}(x)\leqslant Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 3.2. *([3]) If* $\gamma\in R$, *then*

$$\sum_{k=0}^{n}|k-nx|^\gamma p_{n,k}(x)\leqslant Cn^{\frac{\gamma}{2}}\varphi^\gamma(x). \tag{3.2}$$

Lemma 3.3. *For any* $f\in W^r_{w,\lambda}$, $0\leqslant\lambda\leqslant 1$ *and* $\alpha,\ \beta>0$, *we have*

$$\|w\varphi^{r\lambda}F^{(r)}_n\|\leqslant C\|w\varphi^{r\lambda}f^{(r)}\|. \tag{3.3}$$

Proof. By symmetry, we only prove the above result when $x\in(0,1/2]$, the others can be done similarly. Obviously, when $x\in(0,1/n]$, by [5], we have

$$\begin{aligned}|L^{(r)}_r(f,x)|\leqslant C|\overrightarrow{\Delta}^r_{\frac{1}{r}}f(0)|\ &\leqslant\ Cn^{-\frac{r}{2}+1}\int_0^{\frac{r}{n}}u^{\frac{r}{2}}|f^{(r)}(u)|du\\ &\leqslant Cn^{-\frac{r}{2}+1}\|w\varphi^{r\lambda}f^{(r)}\|\int_0^{\frac{r}{n}}u^{\frac{r}{2}}w^{-1}(u)\varphi^{-r\lambda}(u)du.\end{aligned}$$

So

$$|w(x)\varphi^{r\lambda}(x)F^{(r)}_n(x)|\leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

If $x\in[\frac{1}{n},\frac{2}{n}]$, we have

$$\begin{aligned}|w(x)\varphi^{r\lambda}(x)F^{(r)}_n(x)|\leqslant|w(x)\varphi^{r\lambda}(x)f^{(r)}(x)|+|w(x)\varphi^{r\lambda}(x)(f(x)-F_n(x))^{(r)}|\\ :=I_1+I_2.\end{aligned}$$

For I_2, we have

$$f(x)-F_n(x)=(\psi(nx-1)+1)(f(x)-L_r(f,x)).$$

$$w(x)\varphi^{r\lambda}(x)|(f(x)-F_n(x))^{(r)}|=w(x)\varphi^{r\lambda}(x)\sum_{i=0}^{r}n^i|(f(x)-L_r(f,x))^{(r-i)}|.$$

By [5], then

$$|(f(x)-L_r(f,x))^{(r-i)}|_{[\frac{1}{n},\frac{2}{n}]}\leqslant C(n^{r-i}\|f-L_r\|_{[\frac{1}{n},\frac{2}{n}]}+n^{-i}\|f^{(r)}\|_{[\frac{1}{n},\frac{2}{n}]}),\ 0<j<r.$$

Now, we estimate

$$I:=w(x)\varphi^{r\lambda}(x)|f(x)-L_r(x)|. \tag{3.4}$$

By Taylor expansion, we have

$$f(\frac{i}{n})=\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)+\frac{1}{(r-1)!}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds, \tag{3.5}$$

It follows from (3.5) and the identity

$$\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x)=Cx^v,\ v=0,1,\cdots,r.$$

We have

$$\begin{aligned}L_r(f,x)&=\sum_{i=1}^{r}\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)l_i(x)+\frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds\\&=f(x)+C\sum_{u=1}^{r-1}f^{(u)}(x)(\sum_{v=0}^{u}C_u^v(-x)^{u-v}\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x))\\&\quad+\ \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,\end{aligned}$$

which implies that

$$w(x)\varphi^{r\lambda}(x)|f(x)-L_r(f,x)|=\frac{1}{r!}w(x)\varphi^{r\lambda}(x)\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,$$

since $|l_i(x)|\leqslant C$ for $x\in[0,\frac{2}{n}],\ i=1,2,\cdots,r$. It follows from $\frac{|\frac{i}{n}-s|^{r-1}}{w(s)}\leqslant\frac{|\frac{i}{n}-x|^{r-1}}{w(x)}$, s between $\frac{i}{n}$ and x, then

$$\begin{aligned}w(x)\varphi^{r\lambda}(x)|f(x)-L_r(f,x)|&\leqslant Cw(x)\varphi^{r\lambda}(x)\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}|f^{(r)}(s)|ds\\&\leqslant C\varphi^{r\lambda}(x)\|w\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r\lambda}(s)ds\\&\leqslant\frac{C}{n^r}\|w\varphi^{r\lambda}f^{(r)}\|.\end{aligned}$$

Thus

$$I\leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

So, we get

$$I_2\leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

Above all, we have

$$|w(x)\varphi^{r\lambda}(x)F_n^{(r)}(x)|\leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.$$

□

Lemma 3.4. *If* $f\in W^r_{w,\lambda}$, $0\leqslant\lambda\leqslant1$ *and* $\alpha,\ \beta>0$, *then*

$$|w(x)(f(x)-L_r(f,x))|_{[0,\frac{2}{n}]}\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}f^{(r)}\|. \tag{3.6}$$

$$|w(x)(f(x)-R_r(f,x))|_{[1-\frac{2}{n},1]}\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}f^{(r)}\|. \tag{3.7}$$

Proof. By Taylor expansion, we have

$$f(\frac{i}{n})=\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)+\frac{1}{r!}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds, \tag{3.8}$$

It follows from (3.8) and the identity

$$\sum_{i=1}^{r-1}(\frac{i}{n})^v l_i(x)=Cx^v,\ \ v=0,1,\dots,r.$$

We have

$$\begin{aligned}L_r(f,x)&=\sum_{i=1}^{r}\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)l_i(x)+\frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds\\&=f(x)+C\sum_{u=1}^{r-1}f^{(u)}(x)(\sum_{v=0}^{u}C_u^v(-x)^{u-v}\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x))\\&\quad+\ \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,\end{aligned}$$

which implies that

$$w(x)|f(x)-L_r(f,x)|=\frac{1}{(r-1)!}w(x)\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,$$

since $|l_i(x)|\leqslant C$ for $x\in[0,\frac{2}{n}],\ i=1,2,\cdots,r$.

It follows from $\frac{|\frac{i}{n}-s|^{r-1}}{w(s)}\leqslant\frac{|\frac{i}{n}-x|^{r-1}}{w(x)}$, s between $\frac{i}{n}$ and x, then

$$\begin{aligned}w(x)|f(x)-L_r(f,x)|&\leqslant Cw(x)\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}|f^{(r)}(s)|ds\\&\leqslant C\frac{\varphi^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r}(s)ds\\&\leqslant C\frac{\delta_n^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r}(s)ds\\&\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|w\varphi^{r\lambda}f^{(r)}\|.\end{aligned}$$

The proof of (3.7) can be done similarly. □

Lemma 3.5. *([11]) For every $\alpha,\ \beta>0$, we have*

$$\|wB^*_{n,r-1}(f)\|\leqslant C\|wf\|. \tag{3.9}$$

Lemma 3.6. *([15]) If $\varphi(x)=\sqrt{x(1-x)},\ 0\leqslant\lambda\leqslant1,\ 0\leqslant\beta\leqslant1$, then*

$$\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\varphi^{-r\beta}(x+\sum_{k=1}^{r}u_k)du_1\cdots du_r\leqslant Ch^r\varphi^{r(\lambda-\beta)}(x). \tag{3.10}$$

4. Proof of Theorems

4.1. **Proof of Theorem 2.1.** By symmetry, in what follows we will always assume that $x\in(0,\frac{1}{2}]$. It is sufficient to prove the validity for $B_{n,r-1}(F_n,x)$ instead of $B^*_{n,r-1}(f,x)$. When $x\in(0,\frac{1}{n})$, we have

$$\begin{aligned}
|w(x)B^{*(r)}_{n,r-1}(f,x)| &\leqslant w(x)\sum_{i=0}^{r-2}\frac{n_i!}{(n_i-r)!}\sum_{k=0}^{n_i-r}|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})|p_{n_i-r,k}(x)\\
&\leqslant Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})|p_{n_i-r,k}(x)\\
&\leqslant Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}\sum_{j=0}^{r}C_r^j|F_n(\frac{k+r-j}{n_i})|p_{n_i-r,k}(x)\\
&\leqslant Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{j=0}^{r}C_r^j|F_n(\frac{r-j}{n_i})|p_{n_i-r,0}(x)\\
&\quad+Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{j=0}^{r}C_r^j|F_n(\frac{n_i-j}{n_i})|p_{n_i-r,n_i-r}(x)\\
&\quad+Cw(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\sum_{j=0}^{r}C_r^j|F_n(\frac{k+r-j}{n_i})|p_{n_i-r,k}(x)\\
&:=H_1+H_2+H_3.
\end{aligned}$$

We have

$$\begin{aligned}
H_1 &\leqslant Cw(x)\|wf\|\sum_{i=0}^{r-2}n_i^rw^{-1}(\frac{1}{n_i})p_{n_i-r,0}(x)\\
&\leqslant C\|wf\|\sum_{i=0}^{r-2}n_i^r(n_ix)^{\alpha}(1-x)^{n_i-r}\\
&\leqslant Cn^r\|wf\|.
\end{aligned}$$

When $1\leqslant k\leqslant n_i-r-1$, we have $1\leqslant k+2r-j\leqslant n_i-1$, and thus

$$\frac{w(\frac{k}{n_i-r})}{w(\frac{k+r-j}{n_i})}=(\frac{n_i}{n_i-r})^{\alpha+\beta}(\frac{k}{k+r-j})^{\alpha}(\frac{n_i-r-k}{n_i-r-k+j})^{\beta}\leqslant C.$$

Thereof, by (3.1), we have

$$\begin{aligned}
H_3 &\leqslant Cw(x)\|wF_n\|\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\sum_{j=0}^{r}\frac{1}{w(\frac{k+r-j}{n_i})}p_{n_i-r,k}(x)\\
&\leqslant Cw(x)\|wF_n\|\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\frac{1}{w(\frac{k}{n_i-r})}p_{n_i-r,k}(x)\\
&\leqslant Cn^r\|wF_n\|\ \leqslant\ Cn^r\|wf\|.
\end{aligned}$$

Similarly, we can get $H_2\leqslant Cn^r\|wf\|$. So

$$|w(x)B^{*(r)}_{n,r-1}(f,x)|\leqslant Cn^r\|wf\|,\ x\in(0,\frac{1}{n}). \tag{4.1}$$

When $x \in [\frac{1}{n}, \frac{1}{2}]$, according to [5], we have

$$\begin{aligned} &|w(x)B^{*(r)}_{n,r-1}(f,x)| \\ &= |w(x)B^{(r)}_{n,r-1}(F_n,x)| \\ &\leqslant w(x)(\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)|n_i^j\sum_{k=0}^{n}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x). \end{aligned}$$

Then
$Q_j(x,n_i)=(n_ix(1-x))^{[\frac{r-j}{2}]}$, and $(\varphi^2(x))^{-r}Q_j(x,n_i)n_i^j \leqslant C(n_i/\varphi^2(x))^{\frac{r+j}{2}}$, we have

$$\begin{aligned} |w(x)B^{*(r)}_{n,r-1}(f,x)| &\leqslant Cw(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ &\leqslant C\|wF_n\|w(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}\frac{|x-\frac{k}{n_i}|^j}{w(\frac{k^*}{n_i})}p_{n_i,k}(x), \end{aligned} \tag{4.2}$$

where $k^*=1$ for $k=0$, $k^*=n_i-1$ for $k=n_i$ and $k^*=k$ for $1<k<n_i$. Note that

$$w^2(x)\frac{p_{n_i,0}(x)}{w^2(\frac{1}{n_i})} \leqslant C(n_ix)^{2\alpha}(1-x)^{n_i} \leqslant C,$$

and

$$w^2(x)\frac{p_{n_i,n_i}(x)}{w^2(1-\frac{1}{n_i})} \leqslant Cn_i^{\beta}x^{n_i} \leqslant C\frac{n_i^{\beta}}{2^{n_i}} \leqslant C.$$

By (3.1), we have

$$\sum_{k=0}^{n_i}\frac{1}{w^2(\frac{k^*}{n_i})}p_{n_i,k}(x) \leqslant Cw^{-2}(x). \tag{4.3}$$

Now, applying Cauchy's inequality, by (3.2) and (4.3), we have

$$\begin{aligned} \sum_{k=0}^{n_i}\frac{|x-\frac{k}{n_i}|^j}{w(\frac{k^*}{n_i})}p_{n_i,k}(x) &\leqslant (\sum_{k=0}^{n_i}|x-\frac{k}{n_i}|^{2j}p_{n_i,k}(x))^{1/2}(\sum_{k=0}^{n_i}\frac{1}{w^2(\frac{k^*}{n_i})}p_{n_i,k}(x))^{1/2} \\ &\leqslant Cn_i^{-j/2}\varphi^j(x)w^{-1}(x). \end{aligned}$$

Substituting this to (4.2), we have

$$|w(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^r\|wf\|,\ x\in[\frac{1}{n},\frac{1}{2}]. \tag{4.4}$$

We get Theorem 2.1 by (4.1) and (4.4). □

4.2. **Proof of Theorem 2.2.** (1) When $f \in C_w$, we proceed it as follows:

Case 1. If $0 \leqslant \varphi(x) \leqslant \frac{1}{\sqrt{n}}$, by (2.2), we have

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^{-r\lambda/2}|w(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant Cn^{r(1-\lambda/2)}\|wf\|. \tag{4.5}$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}|B^{*(r)}_{n,r-1}(f,x)| &= |B^{(r)}_{n,r-1}(F_n,x)| \\ \leqslant (\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{k=0}^{n_i}&|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x),\end{aligned}$$

$Q_j(x,n_i) = (n_ix(1-x))^{[\frac{r-j}{2}]}$, and $(\varphi^2(x))^{-r}Q_j(x,n_i)n_i^j \leqslant C(n_i/\varphi^2(x))^{\frac{r+j}{2}}$.
So

$$\begin{aligned}&|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \\ \leqslant Cw(x)\varphi^{r\lambda}(x)&\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ &\leqslant Cn^{\frac{r}{2}}\varphi^{r(\lambda-1)}(x). \qquad (4.6)\end{aligned}$$

It follows from combining with (4.5) and (4.6) that the first inequality is proved.
(2) When $f \in W^r_{w,\lambda}$, we have

$$B^{(r)}_{n,r-1}(F_n,x) = \sum_{i=0}^{r-2}C_i(n)n_i^r\sum_{k=0}^{n_i-r}\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})p_{n_i-r,k}(x). \qquad (4.7)$$

If $0 < k < n_i - r$, we have

$$|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})| \leqslant Cn_i^{-r+1}\int_0^{\frac{r}{n_i}}|F_n^{(r)}(\frac{k}{n_i}+u)|du, \qquad (4.8)$$

If $k = 0$, we have

$$|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(0)| \leqslant C\int_0^{\frac{r}{n_i}}u^{r-1}|F_n^{(r)}(u)|du, \qquad (4.9)$$

Similarly

$$|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{n_i-r}{n_i})| \leqslant Cn_i^{-r+1}\int_{1-\frac{r}{n_i}}^1(1-u)^{\frac{r}{2}}|F_n^{(r)}(u)|du. \qquad (4.10)$$

By (4.7)-(4.10), we have

$$\begin{aligned}&|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \\ \leqslant Cw(x)\varphi^{r\lambda}(x)\|w\varphi^{r\lambda}F_n^{(r)}\|&\sum_{i=0}^{r-2}\sum_{k=0}^{n_i-r}(w\varphi^{r\lambda})^{-1}(\frac{k^*}{n_i})p_{n_i-r,k}(x), \qquad (4.11)\end{aligned}$$

where $k^* = 1$ for $k = 0$, $k^* = n_i - r - 1$ for $k = n_i - r$ and $k^* = k$ for $1 < k < n_i - r$. By (3.1), we have

$$\sum_{k=0}^{n_i-r}(w\varphi^{r\lambda})^{-1}(\frac{k^*}{n_i})p_{n_i-r,k}(x) \leqslant C(w\varphi^{r\lambda})^{-1}(x). \qquad (4.12)$$

which combining with (4.12) give

$$|w(x)\varphi^{r\lambda}(x)B^{*(r)}_{n,r-1}(f,x)| \leqslant C\|w\varphi^{r\lambda}f^{(r)}\|.\Box$$

So we get the second inequality of the Theorem 2.2.

4.3. **Proof of Theorem 2.3.**

4.3.1. *The direct theorem.* We know

$$F_n(t) = F_n(x) + F_n'(t)(t-x) + \cdots + \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1}F_n^{(r)}(u)du, \tag{4.13}$$

$$B_{n,r-1}((\cdot - x)^k, x) = 0,\ \ k = 1, 2, \cdots, r-1. \tag{4.14}$$

According to the definition of $W^r_{w,\lambda}$, for any $g \in W^r_{w,\lambda}$, we have $B^*_{n,r-1}(g,x) = B_{n,r-1}(G_n(g),x)$, and $w(x)|G_n(x) - B_{n,r-1}(G_n,x)| = w(x)|B_{n,r-1}(R_r(G_n,t,x),x)|$, thereof $R_r(G_n,t,x) = \int_x^t (t-u)^{r-1}G_n^{(r)}(u)du$. It follows from $\frac{|t-u|^{r-1}}{w(u)} \leqslant \frac{|t-x|^{r-1}}{w(x)}$, u between t and x, we have

$$\begin{aligned} w(x)|G_n(x) - B_{n,r-1}(G_n,x)| &\leqslant C\|w\varphi^{r\lambda}G_n^{(r)}\| w(x)B_{n,r-1}\Big(\int_x^t \frac{|t-u|^{r-1}}{w(u)\varphi^{r\lambda}(u)}du, x\Big) \\ &\leqslant C\|w\varphi^{r\lambda}G_n^{(r)}\| w(x)\Big(B_{n,r-1}\Big(\int_x^t \frac{|t-u|^{r-1}}{\varphi^{2r\lambda}(u)}du, x\Big)\Big)^{\frac{1}{2}} \cdot \\ &\quad \Big(B_{n,r-1}\Big(\int_x^t \frac{|t-u|^{r-1}}{w^2(u)}du, x\Big)\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.15}$$

also

$$\int_x^t \frac{|t-u|^{r-1}}{\varphi^{2r\lambda}(u)}du \leqslant C\frac{|t-x|^r}{\varphi^{2r\lambda}(x)},\ \ \int_x^t \frac{|t-u|^{r-1}}{w^2(u)}du \leqslant \frac{|t-x|^r}{w^2(x)}. \tag{4.16}$$

By (3.2), (4.15) and (4.16), we have

$$\begin{aligned} w(x)|G_n(x) - B_{n,r-1}(G_n,x)| &\leqslant C\|w\varphi^{r\lambda}G_n^{(r)}\|\varphi^{-r\lambda}(x)B_{n,r-1}(|t-x|^r, x) \\ &\leqslant Cn^{-\frac{r}{2}}\frac{\varphi^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}G_n^{(r)}\| \\ &\leqslant Cn^{-\frac{r}{2}}\frac{\delta_n^r(x)}{\varphi^{r\lambda}(x)}\|w\varphi^{r\lambda}G_n^{(r)}\| \\ &= C\Big(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)}\Big)^r\|w\varphi^{r\lambda}G_n^{(r)}\|. \end{aligned} \tag{4.17}$$

By (3.3), (3.6), (3.7) and (4.17), when $g \in W^r_{w,\lambda}$, then

$$\begin{aligned} w(x)|g(x) - B^*_{n,r-1}(g,x)| &\leqslant w(x)|g(x) - G_n(g,x)| + w(x)|G_n(g,x) - B^*_{n,r-1}(g,x)| \\ &\leqslant |w(x)(g(x) - L_r(g,x))|_{[0,\frac{2}{n}]} + |w(x)(g(x) - R_r(g,x))|_{[1-\frac{2}{n},1]} \\ &\quad + C\Big(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)}\Big)^r\|w\varphi^{r\lambda}G_n^{(r)}\| \\ &\leqslant C\Big(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)}\Big)^r\|w\varphi^{r\lambda}g^{(r)}\|. \end{aligned} \tag{4.18}$$

For $f \in C_w$, we choose proper $g \in W^r_{w,\lambda}$, by (3.9) and (4.18), then

$$\begin{aligned} w(x)|f(x) - B^*_{n,r-1}(f,x)| &\leqslant w(x)|f(x) - g(x)| + w(x)|B^*_{n,r-1}(f-g,x)| \\ &\quad + w(x)|g(x) - B^*_{n,r-1}(g,x)| \\ &\leqslant C\Big(\|w(f-g)\| + \Big(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)}\Big)^r\|w\varphi^{r\lambda}g^{(r)}\|\Big) \\ &\leqslant C\omega^r_{\varphi^{\lambda}}\Big(f, \frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)}\Big)_w. \square \end{aligned}$$

4.3.2. *The inverse theorem.* We define the weighted main-part modulus for $D = R_+$ by

$$\Omega^r_{\varphi^\lambda}(C,f,t)_w = \sup_{0<h\leqslant t} \|w\Delta^r_{h\varphi^\lambda} f\|_{[Ch^*,\infty]},$$
$$\Omega^r_{\varphi^\lambda}(1,f,t)_w = \Omega^r_{\varphi^\lambda}(f,t)_w.$$

where $C > 2^{1/\beta(0)-1}$, $\beta(0) > 0$ and h^* is given by

$$h^* = \begin{cases} (Ar)^{1/1-\beta(0)} h^{1/1-\beta(0)}, & 0 \leqslant \beta(0) < 1, \\ 0, & \beta(0) \geqslant 1. \end{cases}$$

The main-part K-functional is given by

$$K_{r,\varphi^\lambda}(f,t^r)_w = \sup_{0<h\leqslant t} \inf_g \{\|w(f-g)\|_{[Ch^*,\infty]} + t^r \|w\varphi^{r\lambda} g^{(r)}\|_{[Ch^*,\infty]},$$

where $g^{(r-1)} \in A.C.((Ch^*,\infty))\}$. By ([5]), we have

$$C^{-1}\Omega^r_{\varphi^\lambda}(f,t)_w \leqslant \omega^r_{\varphi^\lambda}(f,t)_w \leqslant C\int_0^t \frac{\Omega^r_{\varphi^\lambda}(f,\tau)_w}{\tau} d\tau, \tag{4.19}$$

$$C^{-1}K_{r,\varphi^\lambda}(f,t^r)_w \leqslant \Omega^r_{\varphi^\lambda}(f,t)_w \leqslant CK_{r,\varphi^\lambda}(f,t^r)_w. \tag{4.20}$$

Proof. Let $\delta > 0$, by (4.20), we choose proper g so that

$$\|w(f-g)\| \leqslant C\Omega^r_{\varphi^\lambda}(f,\delta)_w,\ \|w\varphi^{r\lambda}g^{(r)}\| \leqslant C\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w. \tag{4.21}$$

then

$$\begin{aligned}
|w(x)\Delta^r_{h\varphi^\lambda} f(x)| &\leqslant |w(x)\Delta^r_{h\varphi^\lambda}(f(x) - B^*_{n,r-1}(f,x))| + |w(x)\Delta^r_{h\varphi^\lambda} B^*_{n,r-1}(f-g,x)| \\
&\quad + |w(x)\Delta^r_{h\varphi^\lambda} B^*_{n,r-1}(g,x)| \\
&\leqslant \sum_{j=0}^r C^j_r \Big(n^{-\frac12} \frac{\delta_n(x+(\frac r2 - j)h\varphi^\lambda(x))}{\varphi^\lambda(x+(\frac r2 - j)h\varphi^\lambda(x))}\Big)^{\alpha_0} \\
&\quad + \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} w(x) B^{*(r)}_{n,r-1}(f-g, x+\sum_{k=1}^r u_k) du_1 \cdots du_r \\
&\quad + \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} w(x) B^{*(r)}_{n,r-1}(g, x+\sum_{k=1}^r u_k) du_1 \cdots du_r \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{4.22}$$

Obviously

$$J_1 \leqslant C(n^{-\frac12}\varphi^{-\lambda}(x)\delta_n(x))^{\alpha_0}. \tag{4.23}$$

By (2.2) and (4.21), we have

$$\begin{aligned}
J_2 &\leqslant Cn^r \|w(f-g)\| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} du_1 \cdots du_r \\
&\leqslant Cn^r h^r \varphi^{r\lambda}(x) \|w(f-g)\| \\
&\leqslant Cn^r h^r \varphi^{r\lambda}(x) \Omega^r_{\varphi^\lambda}(f,\delta)_w.
\end{aligned} \tag{4.24}$$

By the first inequality of (2.3), we let $\lambda = 1$, and (3.10) as well as (4.21), we have

$$\begin{aligned} J_2 &\leqslant Cn^{\frac{r}{2}}\|w(f-g)\| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-r}(x+\sum_{k=1}^{r} u_k) du_1 \cdots du_r \\ &\leqslant Cn^{\frac{r}{2}} h^r \varphi^{r(\lambda-1)}(x)\|w(f-g)\| \\ &\leqslant Cn^{\frac{r}{2}} h^r \varphi^{r(\lambda-1)}(x)\Omega^r_{\varphi^\lambda}(f,\delta)_w. \end{aligned} \tag{4.25}$$

By the second inequality of (2.3) and (4.21), we have

$$\begin{aligned} J_3 &\leqslant C\|w\varphi^{r\lambda}g^{(r)}\|w(x) \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} w^{-1}(x+\sum_{k=1}^{r} u_k)\varphi^{-r\lambda}(x+\sum_{k=1}^{r} u_k) du_1 \cdots du_r \\ &\leqslant Ch^r\|w\varphi^{r\lambda}g^{(r)}\| \\ &\leqslant Ch^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w. \end{aligned} \tag{4.26}$$

Now, by (4.22)-(4.26), we get

$$|w(x)\Delta^r_{h\varphi^\lambda} f(x)| \leqslant C\{(n^{-\frac{1}{2}}\delta_n(x))^{\alpha_0} + h^r(n^{-\frac{1}{2}}\delta_n(x))^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w + h^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w\}.$$

When $n \geqslant 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leqslant \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper $x, n \in N$, so that

$$n^{-\frac{1}{2}}\delta_n(x) \leqslant \delta < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x),$$

Therefore

$$|w(x)\Delta^r_{h\varphi^\lambda} f(x)| \leqslant C\{\delta^{\alpha_0} + h^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_w\}.$$

By Borens-Lorentz lemma in [5], we get

$$\Omega^r_{\varphi^\lambda}(f,t)_w \leqslant Ct^{\alpha_0}. \tag{4.27}$$

So, by (4.27), we get

$$\omega^r_{\varphi^\lambda}(f,t)_w \leqslant C\int_0^t \frac{\Omega^r_{\varphi^\lambda}(f,\tau)_w}{\tau} d\tau = C\int_0^t \tau^{\alpha_0-1} d\tau = Ct^{\alpha_0}.$$

□

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